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Higgs production and decay: analytic results at next-to-leading order QCD

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Abstract: The virtual two-loop corrections for Higgs production in gluon fusion are calculated analytically in QCD for arbitrary Higgs and quark masses. Both scalar and pseudo-scalar Higgs bosons are considered. The results are obtained by expanding the known one-dimensional integral representation in terms of m_H/m_q, and matching it with a suitably chosen ansatz of Harmonic Polylogarithms. This ansatz is motivated by the known analytic result for the Higgs decay rate into two photons. The method also allows us to check this result and to extend it to the pseudo-scalar decay rate.

Keywords: NLO Computations, Higgs Physics, Hadronic Colliders.

Contents

1. Introduction

The gluon fusion process for Higgs production at a hadron collider has been studied in great detail over the last few years (for a recent review, see refs. [1, 2]). It is well-known to be the dominant mode in the Standard Model and also in most of the usually considered supersymmetric parameter space. The fact that the next-to-leading order (NLO) QCD corrections [3 – 5] increase the cross section by more than 70% triggered more detailed studies of higher order effects. In particular, the NNLO [6 – 10] and quite recently even the leading threshold-enhanced N^{3}LO [11] corrections were evaluated in the heavy-top limit, indicating a well-behaved perturbative expansion of the total cross section. Meanwhile, the NNLO effects are known also for differential quantities in terms of a partonic NNLO Monte Carlo program [12, 13], allowing to simulate experimental cuts, for example.

Concerning the Higgs decay rate into photons, the QCD corrections are known through α_s [14 – 21]. For a scalar Higgs boson, the $\mathcal{O}(\alpha_s^2)$ corrections have been evaluated in terms of an expansion around $m_H/m_t = 0$ [22]. In the scalar case, also the leading electroweak effects are known, both for the photonic decay rate [23 – 25] and the gluon fusion process [26, 23, 27].

In contrast to the NNLO calculations which currently all rely on the heavy-top limit, the inclusive NLO effects were calculated for arbitrary values of the Higgs boson mass and the mass of the quark that mediates the gluon-Higgs or photon-Higgs coupling [28, 5, 29, 21]. In fact, it is these calculations that justify the use of the heavy-top limit at NNLO, because they explicitely demonstrate the excellent quality of this limit even at Higgs masses close to the quark threshold $m_H \approx 2m_q$ and beyond. Probably the most important application of the general m_H/m_q dependence currently is supersymmetry, where bottom quarks can contribute significantly to the gluon-Higgs and photon-Higgs coupling due to a potential enhancement proportional to $\tan\beta$ of their Yukawa coupling to Higgs bosons. For bottom quarks, an analogous "heavy-quark" approximation would certainly be very doubtful in this context [5, 30, 29, 31].

The result for the full mass dependence of ref. [28, 5] has been evaluated in terms of an intricate one-dimensional integral representation and implemented in the FORTRAN routine HIGLU [32]. In addition, analytical formulas for various kinematical limits have been given. Clearly, this has been sufficient for numerous phenomenological applications.

On the other hand, portability, manageability, or the possibility to derive any analytical approximation in a simple way, are among the virtues that may make an analytical result desirable. Not to forget, the technical aspects of a closed analytical result — in particular the way it was derived — are usually issues of interest by themselves.

Technical progress concerning the analytic evaluation of multi-loop Feynman diagrams has been enourmous in the last decade. For example, the analytic results for all (scalar) two-loop integrals related to the processes discussed here, are meanwhile available in the literature. In fact, this class of Feynman integrals currently receives great attention due to its importance for electro-weak precision observables (see, e.g., refs. [33 – 35] and references therein).

In this paper we derive the closed analytic expression for the two-loop virtual terms corresponding to the processes $H/A \to \gamma\gamma$ and $gg \to H/A$. Let us stress that we did not evaluate the corresponding Feynman diagrams; rather, we used the integral representation given in ref. [5] and evaluated it analytically. The method we followed is rather unconventional but not new. For the sake of brevity, we will refer to it as *Expansion and Inversion (E&I)* in what follows. It relies on the identity theorem for power series: Two analytic functions are the same if their Taylor series are the same.[1] A more detailed description of the method and its realization will be given in section 2.

2. Discussion of the method; calculation of the decay rates

The idea behind our approach is that, if two physical processes correspond to a similar set of Feynman diagrams (kinematics, mass assignment), their cross sections should be described by a common set of analytical functions. Thus, if one processes is known, one can establish an ansatz for the other one by a linear combination of these functions, with unknown coefficients. In the E&I method, one then evaluates the power series of the unknown cross section in a certain limit and compares it with the corresponding expansion of the ansatz. This leads to a system of linear equations for the unknown coefficients which can be solved uniquely if the depth of the expansion matches the number of unknowns. In general, it is advisable to overdetermine the system in order to confirm that the ansatz is complete. The identity theorem for power series ensures that the solution obtained in this way is indeed the analytical result for the cross section.

[1] The functions related to Feynman diagrams typically fulfill all the requirements that are needed for the precise form of this theorem: The radius of convergence is non-zero, and the limit of the power series is the function itself.

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It is important to realize that, while the intermediate power series approximates the full result only within the radius of convergence, the final result is valid for arbitrary values of the parameters. Thus, the comparison of the final result with a numerical evaluation of the original integral *outside* the radius of convergence provides one of the most powerful checks on the calculation.

The main advantage of the E&I approach is that, in most cases, the power series of the cross section can be obtained in a rather simple manner. A powerful tool for this goal is provided by asymptotic expansions of Feynman diagrams (see refs. [36 – 38] and references therein). This method works directly at the level of Feynman integrals and has been fully automated for the case of euclidean external momenta [39, 40].

Very often, however, one can derive one-dimensional integral representations over finite integration regions. This can be achieved by introducing Feynman parameters, for example, and performing all but one integral analytically. If the interchange of integration and differentiation is possible, the power series expansion can be performed directly on the integrand, and the resulting integrals are in general much simpler than the original ones.

Another example where the integrations are over finite regions is given by phase space integrals, and in fact, the E&I method has been used for the evaluation of the three-particle phase space integration in the case of Higgs production and the Drell-Yan process, both at NNLO [6, 8, 41, 42].

In this paper, we apply the E&I method to obtain analytic formulae for the NLO predictions of the Higgs decay rate into two photons, as well as to the virtual two-loop corrections for Higgs production in gluon fusion. We consider both scalar and pseudo-scalar Higgs bosons such that our results are relevant also in supersymmetric scenarios or other extensions of the Standard Model.

For all these quantities, a one-dimensional integral representation is known [5]. By interchanging differentiation and integration, we can derive their power series in terms of m_H/m_q. A closed analytical result is only known for the NLO decay rate of a scalar Higgs boson into photons [21]. We use it as the main motivation of our ansatz in order to derive closed analytical expressions for the other quantities as well. It will be useful for the rest of this paper to quote the explicit result at this point.

2.1 Decay rate $H \to \gamma\gamma$

The decay rate of a Higgs boson into two photons through NLO can be written as (see, e.g., ref. [5])

$$\Gamma(H \to \gamma\gamma) = \frac{G_F \alpha^2 m_H^3}{128\sqrt{2}\pi^3} \left| \sum_l Q_l^2 A_l^H(\tau_l) + 3 \sum_q Q_q^2 A_q^H(\tau_q) + A_W^H(\tau_W) \right|^2 , \tag{2.1}$$

where G_F is the Fermi constant, α is the electromagnetic fine-structure constant, m_H denotes the Higgs mass, and $Q_{q,l}$ the electric charge of quark q and lepton l in units of the proton charge. The variables τ_i are defined as

$$\tau_i := \frac{m_H^2}{4m_i^2} , \tag{2.2}$$

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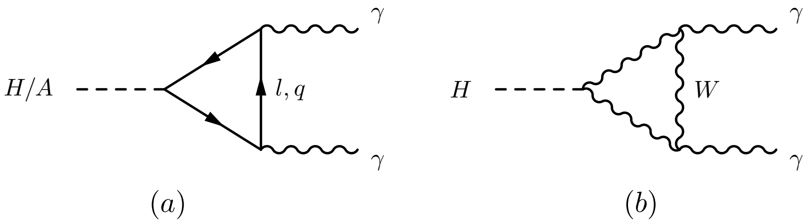

Figure 1: Feynman diagrams contributing to the decay rate for $H/A \to \gamma\gamma$ at leading order.

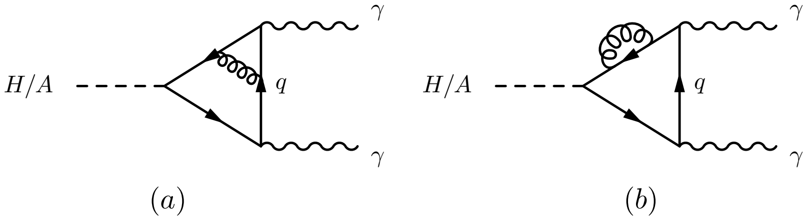

Figure 2: Sample Feynman diagrams contributing to the decay rate for $H/A \to \gamma\gamma$ at NLO.

where m_i denotes the mass of particle i. Here and in what follows, $m_q \equiv m_q(\mu)$ denotes the running quark mass as defined in eq. (5) of ref. [5][2] at the renormalization point μ.

The amplitudes $A_l^H(\tau)$ and $A_W^H(\tau)$ arise from closed lepton and W-boson loops, respectively (cf. figure 1), and do not receive QCD corrections. They are given by [43, 44]

$$\begin{aligned} A_l^H(\tau) &= \frac{2}{\tau^2}\left[\tau + (\tau-1)f(\tau)\right] \equiv \frac{4}{3}\, F_0^H(\tau)\,, \\ A_W^H(\tau) &= -\frac{1}{\tau^2}\left[2\tau^2 + 3\tau + 3(2\tau-2)f(\tau)\right], \end{aligned} \tag{2.3}$$

where

$$f(\tau) = \begin{cases} \arcsin^2\left(\sqrt{\tau}\right), & \tau \le 1 \\ -\dfrac{1}{4}\left[\ln\dfrac{1+\sqrt{1-\tau^{-1}}}{1-\sqrt{1-\tau^{-1}}} - i\pi\right]^2, & \tau > 1\,. \end{cases} \tag{2.4}$$

$A_q^H(\tau)$ originates from the quark mediated photon-Higgs coupling, figure 1$\,(a)$ and figure 2. Through NLO QCD, one can write it as

$$A_q^H(\tau) = \frac{4}{3}\, F_0^H(\tau)\,\left[1 + \frac{\alpha_s}{\pi}\left(C_1^H(\tau) + C_2^H(\tau)\ln\frac{4\tau\mu^2}{m_H^2}\right)\right], \tag{2.5}$$

with $F_0^H(\tau)$ from eq. (2.3). C_2^H follows directly from the NLO renormalization group equation

$$\left(\mu^2\frac{\partial}{\partial\mu^2} + 2\,\frac{\alpha_s}{\pi}\,\tau\frac{\partial}{\partial\tau}\right) A_q^H(\tau) = \mathcal{O}(\alpha_s^2)\,. \tag{2.6}$$

It reads

$$F_0^H\, C_2^H = \frac{3}{\tau^2}\left[\tau + (\tau-2)\, f(\tau) - (\tau-1)\tau\, f'(\tau)\right]. \tag{2.7}$$

[2]Note that m_q is *not* the running $\overline{\text{MS}}$ mass.

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The analytical expression for C_1^H has been obtained in ref. [21]:[3]

$$\begin{aligned}
F_0^H\,C_1^H \;=\; &-\frac{\theta\left(1+\theta+\theta^2+\theta^3\right)}{(1-\theta)^5}\Big[108\,\mathrm{Li}_4(\theta)+144\,\mathrm{Li}_4(-\theta)-64\,\mathrm{Li}_3(\theta)\ln\theta-\\
&\qquad -64\,\mathrm{Li}_3(-\theta)\ln\theta+14\,\mathrm{Li}_2(\theta)\ln^2\theta+8\,\mathrm{Li}_2(-\theta)\ln^2\theta+\\
&\qquad +\frac{1}{12}\ln^4\theta+4\,\zeta_2\ln^2\theta+16\,\zeta_3\ln\theta+18\,\zeta_4\Big]+\\
&+\frac{\theta(1+\theta)^2}{(1-\theta)^4}\Big[-32\,\mathrm{Li}_3(-\theta)+16\,\mathrm{Li}_2(-\theta)\ln\theta-4\,\zeta_2\ln\theta\Big]-\\
&-\frac{4\,\theta\left(7-2\,\theta+7\,\theta^2\right)}{(1-\theta)^4}\,\mathrm{Li}_3(\theta)+\frac{8\,\theta\left(3-2\,\theta+3\,\theta^2\right)}{(1-\theta)^4}\,\mathrm{Li}_2(\theta)\ln\theta+\\
&+\frac{2\,\theta\left(5-6\,\theta+5\,\theta^2\right)}{(1-\theta)^4}\ln(1-\theta)\ln^2\theta+\frac{\theta\left(3+25\,\theta-7\,\theta^2+3\,\theta^3\right)}{3(1-\theta)^5}\ln^3\theta+\\
&+\frac{4\,\theta\left(1-14\,\theta+\theta^2\right)}{(1-\theta)^4}\zeta_3+\frac{12\,\theta^2}{(1-\theta)^4}\ln^2\theta-\frac{12\,\theta\,(1+\theta)}{(1-\theta)^3}\ln\theta-\frac{20\,\theta}{(1-\theta)^2}\,,
\end{aligned}\tag{2.8}$$

with Riemann's zeta function

$$\zeta_n\equiv\zeta(n)\,,\qquad \text{i.e.}\quad \zeta_2=\frac{\pi^2}{6}\,,\quad \zeta_3=1.20206\ldots\,,\quad \zeta_4=\frac{\pi^4}{90}\,,\tag{2.9}$$

and

$$\theta\equiv\theta(\tau)=\frac{\sqrt{1-\tau^{-1}}-1}{\sqrt{1-\tau^{-1}}+1}\,.\tag{2.10}$$

For analytic continuation, it is always understood that $\tau\to\tau+i0$.

The products of logarithms and polylogarithms in eq. (2.8) can be expressed in terms of Harmonic Polylogarithms [45] of the form $\mathrm{H}(\vec{n};\theta)$, where $\vec{n}$ is an n-tuple with entries ("indices") ±1 or 0. One finds that $n\leq4$, and that at most one index is different from zero.

This suggests to construct our ansatz from Harmonic Polylogarithms of this form, multiplied by rational functions

$$R_{l,k}(\theta)=\frac{P_l(\theta)}{(1-\theta)^k}\,,\tag{2.11}$$

where $P_l(\theta)$ is a polynomial in θ of degree l with unknown coefficients. In order to solve the resulting system of linear equations, one has to adjust the integer parameters l,k such that a suitable balance is obtained between the universality of the ansatz and the depth of the power series expansion that is required to determine the unknown coefficients.

[3]Eqs. (10) and (12) of ref. [21] contain typos; thanks to O. Tarasov for immediate confirmation. Note that in order to compare eq. (2.8) with the formula in ref. [21], one needs to use the identity $\mathrm{Li}_3(\theta^2)=4\big[\mathrm{Li}_3(\theta)+\mathrm{Li}_3(-\theta)\big]$.

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As a warm-up, we may try to reproduce eq. (2.8) from the one-dimensional integral representation of ref. [5] using our approach. To this aim, we expand the integrands of $I_1, \ldots, I_5$, defined in eqs. (A.9) to (A.13) of ref. [5],[4] around the limit $\tau = 0$, keeping terms through order τ^{100}.

It is clear that due to the complexity of the integrands and the required depth of the expansion we need to use efficient computer algebra tools. We found that the TAYLOR package [46] for REDUCE [47] is particularly well suited for this kind of operations. In most cases, the results obtained from TAYLOR were checked against our own implementation of the relevant power series in FORM [48]. The capabilities of MATHEMATICA [49], on the other hand, are clearly not suited for expanding expressions of this complexity.[5]

For illustration of the method, let us consider the simplest one of the relevant integrals:

$$I_5 = \int_0^1 \frac{\mathrm{d}x}{1-\rho x} \left\{ \alpha_+ \ln\left(1 - \frac{x}{\alpha_+}\right) + \alpha_- \ln\left(1 - \frac{x}{\alpha_-}\right) \right\} \ln\left(\frac{1-\rho x(1-x)}{x}\right), \tag{2.12}$$

where

$$\rho = 4\tau = \frac{m_H^2}{m_q^2}\,, \qquad \alpha_\pm = \frac{(1 \pm \sqrt{1-\tau^{-1}})}{2}\,.$$

Expanding the integrand around $\tau = 0$ leads to very simple integrations in x,

$$\begin{aligned} I_5 = \int_0^1 \mathrm{d}x \Bigg\{ & 2x \ln x + \tau \left[8x^2 - 8\,x^3 + \ln x \left(10x^2 - \frac{8}{3}\,x^3\right)\right] + \\ & + \tau^2 \left[56\,x^3 - \frac{248}{3}x^4 + \frac{80}{3}x^5 + \ln x \left(\frac{136}{3}x^3 - \frac{68}{3}\,x^4 + \frac{32}{5}\,x^5\right)\right] + \\ & + \tau^3 \Bigg[304\,x^4 - \frac{1744}{3}\,x^5 + \frac{5504}{15}\,x^6 - \frac{448}{5}\,x^7 + \\ & \quad + \ln x \left(\frac{592}{3}x^4 - \frac{2128}{15}x^5 + \frac{1184}{15}x^6 - \frac{128}{7}\,x^7\right)\Bigg] + \\ & + \tau^4 \Bigg[\frac{4480}{3}\,x^5 - \frac{156256}{45}x^6 + \frac{16192}{5}x^7 - \frac{164704}{105}x^8 + \frac{97408}{315}x^9 + \\ & \quad + \ln x \left(\frac{12608}{15}\,x^5 - \frac{3904}{5}\,x^6 + \frac{67712}{105}x^7 - \frac{2080}{7}x^8 + \frac{512}{9}x^9\right)\Bigg]\Bigg\} + \cdots, \end{aligned} \tag{2.13}$$

such that

$$\begin{aligned} I_5 = & -\frac{1}{2} - \frac{5}{18}\tau - \frac{29}{150}\tau^2 - \frac{4882}{33075}\tau^3 - \frac{11786}{99225}\tau^4 - \frac{3564004}{36018675}\tau^5 - \\ & - \frac{95238032}{1127251125}\tau^6 - \frac{745588736}{10145260125}\tau^7 - \frac{190175733376}{2931980176125}\tau^8 + \cdots. \end{aligned} \tag{2.14}$$

[4] Thanks to M. Spira for clarification concerning some typos in the formulas of ref. [5].

[5] We remark that MATHEMATICA 5.1 even produces a wrong result when expanding $\mathrm{Li}_2(1-x)$ around $x = 0$ (also for Li_3 etc.); a bug report has been submitted and acknowledged.

The remaining integrals in ref. [5] are more complex, but the integration of their expansion in τ can always be evaluated in an elementary way. A convenient check of our expansions was provided by ref. [22].

We note in passing that for the coefficient of τ^{100}, as it is required by our general ansatz, the integers in the numerator and the denominator are roughly of order 10^{180}; this should give an idea of the intermediate expressions' complexity.

Nevertheless, this expression, together with the corresponding expansion of the ansatz, can be fed into MATHEMATICA in order to solve the resulting system of linear equations. One finds

$$\begin{aligned} I_5 = \frac{\theta}{(1-\theta)^2}\Big[&4\,\mathrm{Li}_3(\theta) + 8\,\mathrm{Li}_3(-\theta) - 3\,\mathrm{Li}_2(\theta)\ln\theta - 4\,\mathrm{Li}_2(-\theta)\ln\theta - \\ &- \ln(1-\theta)\ln^2\theta + \zeta_2\ln\theta + 2\zeta_3\Big] + \frac{\theta^2}{2(1-\theta)^3}\ln^3\theta. \end{aligned} \tag{2.15}$$

Needless to say that the result obtained for C_1^H in this way is in agreement with eq. (2.8), thus proving the consistency of the analytical result of ref. [21] and the integral representation of ref. [5].

Let us add a few more remarks concerning the construction of the system of linear equations. It happens that the structure of the ansatz can be restricted already from general considerations: The expansions of the entities calculated in this paper all consist only of integer powers of τ multiplied by rational coefficients. The expansions of Harmonic Polylogarithms of argument θ, on the other hand, contain noninteger powers of τ, irrational numbers like ζ_n, logarithms of τ, and have a non-vanishing imaginary part. The fact that such terms do not appear in the expansions of the integrals to be matched produces a lot of equations that constrain our ansatz, regardless of the specific integral to be calculated.

2.2 Decay rate $A \to \gamma\gamma$

After confirming the analytical result for $\Gamma(H \to \gamma\gamma)$ through NLO, we are now ready to apply the method to the pseudo-scalar case. Assuming the Minimal Supersymmetric Standard Model (MSSM) as underlying theory, we write, in analogy to eq. (2.1):

$$\Gamma(A\to\gamma\gamma) = \frac{G_F\alpha^2 m_A^3}{32\sqrt{2}\pi^3}\left|\sum_l Q_l^2 g_l^A A_l^A(\tau_l) + 3\sum_q Q_q^2 g_q^A A_q^A(\tau_q) + \sum_{\tilde{\chi}^\pm} g_{\tilde{\chi}^\pm}^A A_{\tilde{\chi}^\pm}^A(\tau_{\tilde{\chi}^\pm})\right|, \tag{2.16}$$

with the lepton (l) and chargino ($\chi^\pm$) induced amplitudes

$$A_l^A(\tau) = A_{\tilde{\chi}^\pm}^A(\tau) = \frac{f(\tau)}{\tau} \equiv F_0^A(\tau)\,, \tag{2.17}$$

where $f(\tau)$ has been defined in eq. (2.4). Note that there is no contribution from the W as loop particle due to CP invariance. In eq. (2.16), m_A is the mass of the pseudo-scalar Higgs, and the τ-variables are defined according to eq. (2.2), with m_A instead of m_H. The specific values of the couplings $g_{l,q,\tilde{\chi}^\pm}^A$ are irrelevant for our analysis; they can be found in ref. [5].

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In analogy to eq. (2.5), the quark-induced amplitude is written as

$$A_q^A(\tau) = F_0^A(\tau)\left[1+\frac{\alpha_s}{\pi}\left(C_1^A(\tau)+C_2^A(\tau)\ln\frac{4\tau\mu^2}{m_A^2}\right)\right], \tag{2.18}$$

where again C_2^A can be derived through a renormalization group equation analogous to eq. (2.6):

$$F_0^A\,C_2^A = \frac{2}{\tau}\left[f(\tau)-\tau\,f'(\tau)\right]. \tag{2.19}$$

Remarkably, when using the same ansatz as in the scalar case of section 2.1, the resulting system of linear equations has no solution. The necessary generalization is to allow for terms $\sim(1+\theta)^{-1}$ multiplying the Harmonic Polylogarithms, reflecting the well-known threshold singularity in the pseudo-scalar case at $m_A = 2m_q$. Once this is done, the E&I approach yields

$$\begin{aligned}
F_0^A\;C_1^A = &-\frac{\theta\,(1+\theta^2)}{(1-\theta)^3\,(1+\theta)}\Bigg[72\,\mathrm{Li}_4(\theta)+96\,\mathrm{Li}_4(-\theta)-\frac{128}{3}\Big[\mathrm{Li}_3(\theta)+\mathrm{Li}_3(-\theta)\Big]\ln\theta\,+\\
&\qquad+\frac{28}{3}\,\mathrm{Li}_2(\theta)\ln^2\theta+\frac{16}{3}\,\mathrm{Li}_2(-\theta)\ln^2\theta+\frac{1}{18}\ln^4\theta\,+\\
&\qquad+\frac{8}{3}\zeta_2\ln^2\theta+\frac{32}{3}\zeta_3\ln\theta+12\zeta_4\Bigg]+\\
&+\frac{\theta}{(1-\theta)^2}\Bigg[-\frac{56}{3}\,\mathrm{Li}_3(\theta)-\frac{64}{3}\,\mathrm{Li}_3(-\theta)+16\,\mathrm{Li}_2(\theta)\ln\theta\,+\\
&\qquad+\frac{32}{3}\,\mathrm{Li}_2(-\theta)\ln\theta+\frac{20}{3}\ln(1-\theta)\ln^2\theta-\frac{8}{3}\zeta_2\ln\theta+\frac{8}{3}\zeta_3\Bigg]+\\
&+\frac{2\,\theta\,(1+\theta)}{3\,(1-\theta)^3}\,\ln^3\theta\,.
\end{aligned} \tag{2.20}$$

3. Virtual corrections for $gg \to H/A$

An interesting application of our method is the analytical evaluation of the virtual two-loop corrections for Higgs production in gluon fusion for arbitrary values of the quark and Higgs boson mass.

Following ref. [5], we write the inclusive NLO cross section as

$$\sigma(pp\to\Phi+X) = \sigma_0^\Phi\left[1+C^\Phi\,\frac{\alpha_s}{\pi}\right]\tau_\Phi\frac{\mathrm{d}\mathcal{L}^{gg}}{\mathrm{d}\tau_\Phi}+\Delta\sigma_{gg}^\Phi+\Delta\sigma_{gq}^\Phi+\Delta\sigma_{q\bar q}^\Phi\,,\qquad \Phi\in\{H,A\}\,, \tag{3.1}$$

where $\tau_\Phi = m_\Phi^2/s$ with the center-of-mass energy s, and

$$\frac{\mathrm{d}\mathcal{L}^{gg}}{\mathrm{d}\tau} = \int_\tau^1\frac{\mathrm{d}x}{x}g(x,\mu_F)\,g(\tau/x,\mu_F)\,, \tag{3.2}$$

with the gluon density functions $g(x,\mu_F)$, depending on the factorization scale μ_F. The normalization factors are

$$\sigma_0^H = \frac{G_F\alpha_s^2}{288\sqrt{2}\pi}\left|\sum_q g_q^H F_0^H(\tau_q)\right|,\qquad \sigma_0^A = \frac{G_F\alpha_s^2}{128\sqrt{2}\pi}\left|\sum_q g_q^A F_0^A(\tau_q)\right|, \tag{3.3}$$

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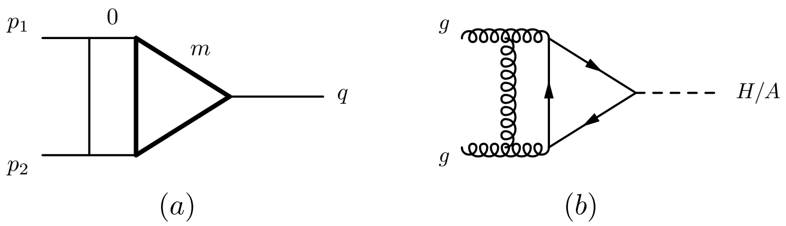

Figure 3: (a) Feynman integral contributing to the production rate $gg \to H/A$ but not to the decay rate $H/A \to \gamma\gamma$; it arises due to the self-coupling of gluons, see (b).

with $F_0^{H/A}$ defined in eqs. (2.3), (2.17). C^Φ denotes the contributions from the virtual two-loop corrections, regularized by the infrared singular part of the cross section for real gluon emission (see ref. [5] for details). It can be decomposed into

$$C^\Phi = \pi^2 + c^\Phi + 2\,\beta_0\, \ln \frac{\mu^2}{m_\Phi^2}\,, \tag{3.4}$$

where $\beta_0 = 11/4 - n_f/6$ is the lowest-order β function of QCD for n_f active quark flavors. The $\Delta\sigma_{ij}^\Phi$ denote the contributions from radiation of quarks and gluons with initial state partons $i, j \in \{q, \bar{q}, g\}$. At NLO perturbation theory, they correspond to massive one-loop three- and four-point functions which can be evaluated analytically using standard techniques [50] (see also ref. [51]). They shall not be considered any further in this paper.

The coefficient c^Φ of the virtual corrections in eq. (3.4) is parameterized as

$$c^\Phi = \mathrm{Re}\left\{ \frac{\sum_q g_q^\Phi F_0^\Phi(\tau_q)\left(B_1^\Phi(\tau_q) + B_2^\Phi(\tau_q)\,\ln\frac{\mu^2}{m_q^2}\right)}{\sum_q g_q^\Phi F_0^\Phi(\tau_q)} \right\}. \tag{3.5}$$

Similar to the decay rates, B_2^Φ follows from renormalization group considerations:

$$B_2^\Phi(\tau) = 2\,C_2^\Phi(\tau)\,, \qquad \Phi \in \{H, A\}\,, \tag{3.6}$$

with C_2^Φ from eqs. (2.7) and (2.19). The factor of 2 arises from the fact that C^Φ in eq. (3.1) is defined at the level of the cross section rather than the amplitude.

B_1^Φ is known again in terms of one-dimensional integrals, filling several pages ($I_1, \ldots, I_8$ in appendix A,B of ref. [5]). Following the method described in section 2, we expanded the integrands for $B_1^\Phi(\tau)$ around $\tau = 0$ up to order τ^{100} and mapped them onto a set of suitably chosen basis functions.

In the case of gluonic Higgs production, a new class of integrals occurs that cannot be expressed in terms of the functions used for the decay rates. The corresponding scalar diagram is shown in figure 3 (a); it arises from the physical process due to the self-interaction of gluons, see figure 3 (b). The scalar integral has been evaluated in ref. [52]. The result contains a Harmonic Polylogarithm of weight four with two indices different from zero. We therefore enlarge our basis to include this kind of functions. Apart from that, the method works exactly like in the case of the decay rates, described in section 2.

With θ defined in eq. (2.10), we find for the scalar case

$$
\begin{aligned}
F_0^H\, B_1^H \;=\; & \frac{\theta(1+\theta)^2}{(1-\theta)^4}\Bigg[72\,\mathrm{H}(1,0,-1,0;\theta)+6\ln(1-\theta)\ln^3\theta-36\,\zeta_2\,\mathrm{Li}_2(\theta)-\\
& -36\,\zeta_2\ln(1-\theta)\ln\theta-108\,\zeta_3\ln(1-\theta)-\\
& -64\,\mathrm{Li}_3(-\theta)+32\,\mathrm{Li}_2(-\theta)\ln\theta-8\,\zeta_2\ln\theta\Bigg]-\\
& -\frac{36\,\theta\left(5+5\,\theta+11\,\theta^2+11\,\theta^3\right)}{(1-\theta)^5}\,\mathrm{Li}_4(-\theta)-\frac{36\,\theta\left(5+5\,\theta+7\,\theta^2+7\,\theta^3\right)}{(1-\theta)^5}\,\mathrm{Li}_4(\theta)+\\
& +\frac{4\theta\,(1+\theta)\left(23+41\theta^2\right)}{(1-\theta)^5}\Big[\mathrm{Li}_3(\theta)+\mathrm{Li}_3(-\theta)\Big]\ln\theta-\\
& -\frac{16\theta\left(1+\theta+\theta^2+\theta^3\right)}{(1-\theta)^5}\,\mathrm{Li}_2(-\theta)\ln^2\theta-\frac{2\theta\left(5+5\theta+23\theta^2+23\theta^3\right)}{(1-\theta)^5}\,\mathrm{Li}_2(\theta)\ln^2\theta+\\
& +\frac{\theta\left(5+5\,\theta-13\,\theta^2-13\,\theta^3\right)}{24(1-\theta)^5}\ln^4\theta+\frac{\theta\left(1+\theta-17\,\theta^2-17\,\theta^3\right)}{(1-\theta)^5}\zeta_2\ln^2\theta+\\
& +\frac{2\,\theta\left(11+11\,\theta-43\,\theta^2-43\theta^3\right)}{(1-\theta)^5}\zeta_3\ln\theta+\frac{36\,\theta\left(1+\theta-3\,\theta^2-3\,\theta^3\right)}{(1-\theta)^5}\zeta_4-\\
& -\frac{2\,\theta\left(55+82\,\theta+55\,\theta^2\right)}{(1-\theta)^4}\,\mathrm{Li}_3(\theta)+\frac{2\,\theta\left(51+74\,\theta+51\,\theta^2\right)}{(1-\theta)^4}\,\mathrm{Li}_2(\theta)\ln\theta+\\
& +\frac{\theta\left(47+66\,\theta+47\,\theta^2\right)}{(1-\theta)^4}\ln(1-\theta)\ln^2\theta+\frac{\theta\left(6+59\,\theta+58\,\theta^2+33\,\theta^3\right)}{3(1-\theta)^5}\ln^3\theta+\\
& +\frac{2\,\theta\left(31+34\,\theta+31\,\theta^2\right)}{(1-\theta)^4}\zeta_3+\frac{3\,\theta\left(3+22\,\theta+3\,\theta^2\right)}{2(1-\theta)^4}\ln^2\theta-\\
& -\frac{24\,\theta\,(1+\theta)}{(1-\theta)^3}\ln\theta-\frac{94\,\theta}{(1-\theta)^2}.
\end{aligned}
\tag{3.7}
$$

For the pseudo-scalar case, we get

$$
\begin{aligned}
F_0^A\, B_1^A \;=\; & \frac{\theta}{(1-\theta)^2}\Bigg[48\,\mathrm{H}(1,0,-1,0;\theta)+4\ln(1-\theta)\ln^3\theta-24\,\zeta_2\,\mathrm{Li}_2(\theta)-\\
& -24\,\zeta_2\ln(1-\theta)\ln\theta-72\,\zeta_3\ln(1-\theta)-\frac{220}{3}\,\mathrm{Li}_3(\theta)-\frac{128}{3}\,\mathrm{Li}_3(-\theta)+\\
& +68\,\mathrm{Li}_2(\theta)\ln\theta+\frac{64}{3}\,\mathrm{Li}_2(-\theta)\ln\theta+\frac{94}{3}\ln(1-\theta)\ln^2\theta-\\
& -\frac{16}{3}\zeta_2\ln\theta+\frac{124}{3}\zeta_3+3\ln^2\theta\Bigg]-\\
& -\frac{24\,\theta\left(5+7\,\theta^2\right)}{(1-\theta)^3\,(1+\theta)}\,\mathrm{Li}_4(\theta)-\frac{24\,\theta\left(5+11\,\theta^2\right)}{(1-\theta)^3\,(1+\theta)}\,\mathrm{Li}_4(-\theta)+\\
& +\frac{8\,\theta\left(23+41\,\theta^2\right)}{3(1-\theta)^3\,(1+\theta)}\Big[\mathrm{Li}_3(\theta)+\mathrm{Li}_3(-\theta)\Big]\ln\theta-\frac{4\,\theta\left(5+23\,\theta^2\right)}{3(1-\theta)^3\,(1+\theta)}\,\mathrm{Li}_2(\theta)\ln^2\theta-\\
& -\frac{32\,\theta\left(1+\theta^2\right)}{3(1-\theta)^3\,(1+\theta)}\,\mathrm{Li}_2(-\theta)\ln^2\theta+\frac{\theta\left(5-13\,\theta^2\right)}{36(1-\theta)^3\,(1+\theta)}\ln^4\theta+
\end{aligned}
$$

$$+\frac{2\,\theta\left(1-17\,\theta^2\right)}{3(1-\theta)^3\,(1+\theta)}\zeta_2\ln^2\theta+\frac{4\,\theta\left(11-43\,\theta^2\right)}{3(1-\theta)^3\,(1+\theta)}\zeta_3\ln\theta+$$
$$+\frac{24\,\theta\left(1-3\,\theta^2\right)}{(1-\theta)^3\,(1+\theta)}\zeta_4+\frac{2\,\theta\,(2+11\,\theta)}{3(1-\theta)^3}\ln^3\theta\,. \tag{3.8}$$

One may notice that the Harmonic Polylogarithm appearing in ref. [52] is different from the one contributing to B_1^H and B_1^A. However, this is only due to an arbitrariness when choosing a basis of Harmonic Polylogarithms. In fact,

$$\begin{aligned}8\,\mathrm{H}(1,0,-1,0;\theta) \;=\; & -8\,\mathrm{H}(-1,0,0,1;-\theta)-2\,\mathrm{S}_{2,2}\big(\theta^2\big)+8\,\mathrm{S}_{2,2}(\theta)+8\,\mathrm{S}_{2,2}(-\theta)+\\ & +4\ln\theta\,\mathrm{S}_{1,2}\big(\theta^2\big)-8\ln\theta\,\mathrm{S}_{1,2}(\theta)-8\ln\theta\,\mathrm{S}_{1,2}(-\theta)-\\ & -8\ln(1-\theta)\,\mathrm{Li}_3(-\theta)+8\ln(1-\theta)\ln\theta\,\mathrm{Li}_2(-\theta)\,,\end{aligned} \tag{3.9}$$

as can be seen using appendix B of ref. [53], for example. Using $\mathrm{H}(1,0,-1,0;\theta)$, all $\mathrm{S}_{2,2}$ and $\mathrm{S}_{1,2}$ cancel in our final result.

4. Numerical results

As already mentioned in section 2, the comparison of the final result with a numerical evaluation of the original integral provides one of the most important checks of the calculation.

The solid and dashed lines of figure 4 (a) and (b) show the real and imaginary part of $C_1^H F_0^H$ and $C_1^A F_0^A$, respectively. The dotted lines in figure 4 show the results obtained from the intermediate power series when keeping terms of order τ^n with $n = 10, 30, 90$. Note that, as expected, the power series does not converge towards the analytic result beyond the radius of convergence, given by $\tau = 1$. In particular, the imaginary part is always zero. Thus, for $\tau \geq 1$, the result arises solely from analytic continuation of the terms reconstructed through E&I.

In addition, we were able to reproduce figures 5 and 18 of ref. [5], using our results, to perfect agreement. We find a similar picture for the virtual corrections to gluon fusion, shown in figure 5 (a) and (b) for the scalar and the pseudo-scalar case, respectively. The numerical evaluation of $\mathrm{H}(1,0,-1,0;\theta)$ was done with the help of the MATHEMATICA file `HPL4.m` [54].

5. Conclusions

The two-loop QCD results for the decay rate of a scalar or pseudo-scalar Higgs boson into photons, $H/A \to \gamma\gamma$, as well as for the virtual corrections to the production modes $gg \to H/A$ were presented in closed analytical form. In order to obtain these results, we first expanded the known one-dimensional integral representations in terms of small Higgs masses, and subsequently mapped this expansion onto a set of analytic functions. The final results, both for their real and imaginary part, are valid for arbitrary values of the quark and Higgs boson mass. They contain only polylogarithms or simpler functions and, in the case of $gg \to H/A$, one Harmonic Polylogarithm.

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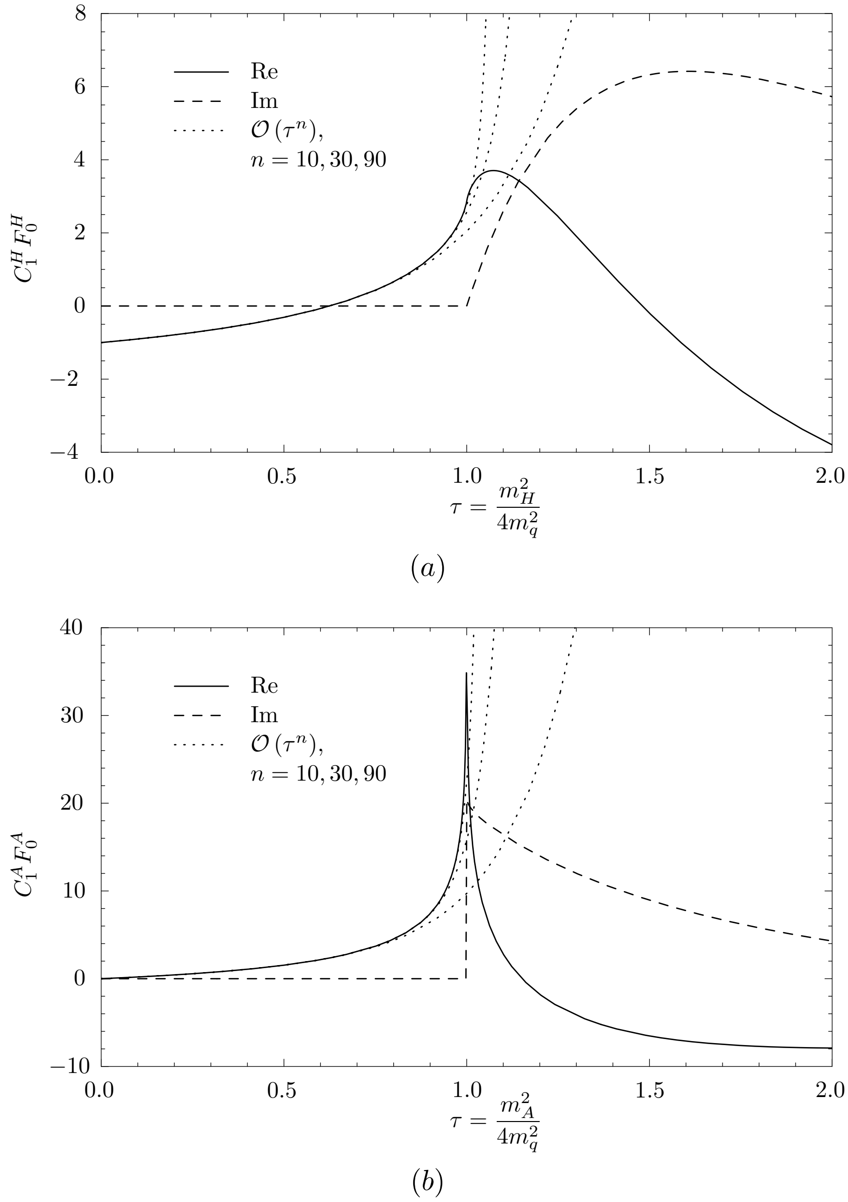

Figure 4: Two-loop contributions to the (a) scalar and (b) pseudo-scalar Higgs decay rate into photons. The solid and dashed lines show the real and the imaginary part, respectively. The dotted lines correspond to the power series expansion up to order τ^n with $n = 10, 30, 90$.

Our formulas should be useful for implementations into physics analysis programs, or for quickly obtaining analytical limits to arbitrary accuracy.

Let us finish by pointing out that the E&I method, in various flavors, has been quite useful already in the past (see refs. [55 − 58, 8, 41], for example). Its combination with asymptotic expansions may even carry the potential for an algorithmic evaluation of Feynman integrals. However, this not only requires much more efficient computer algebra tools for the expansion of Feynman diagrams. The more important task is to find suitable bases for certain classes of Feynman integrals. We believe that this is certainly a task worth pursuing.

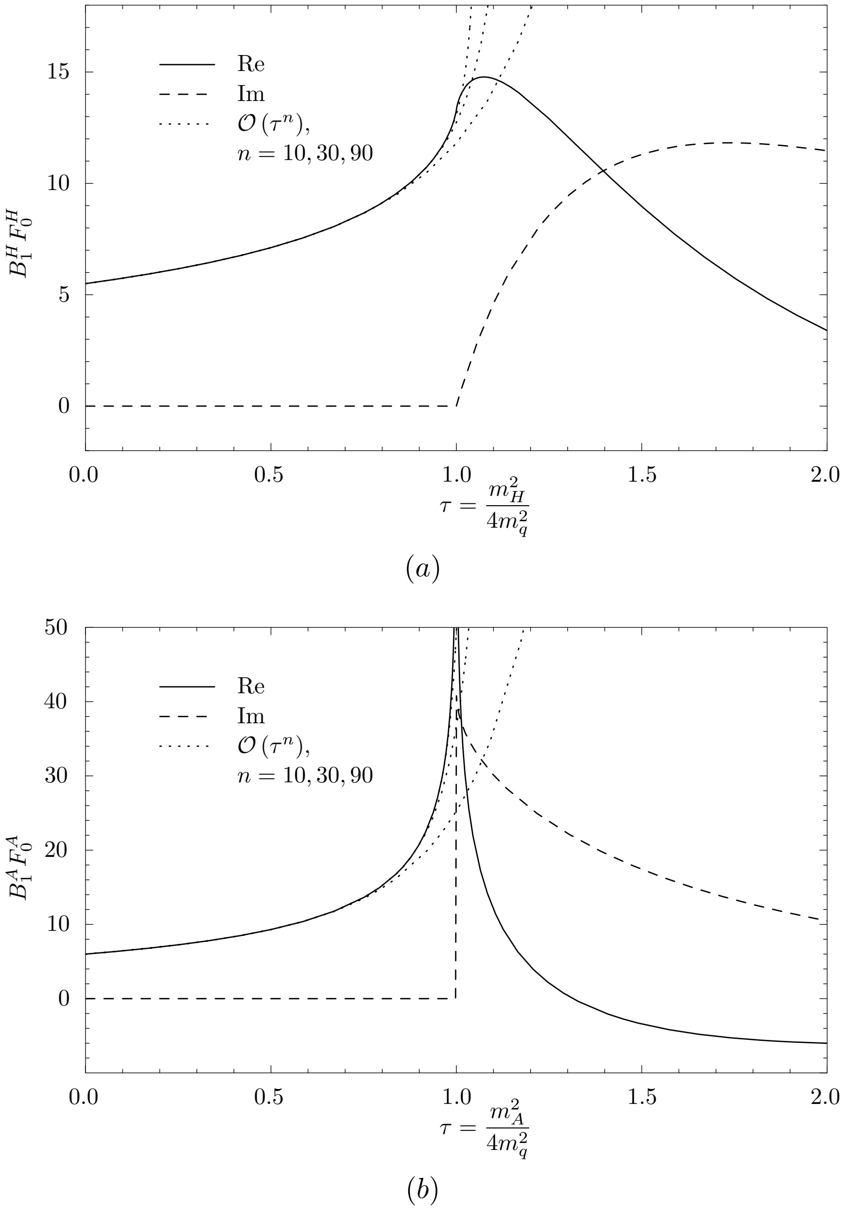

Figure 5: Infrared regularized virtual two-loop corrections to the (a) scalar and (b) pseudo-scalar Higgs production rate through gluon fusion. The solid and dashed lines show the real and the imaginary part, respectively. The dotted lines correspond to the power series expansion up to order τ^n with $n = 10, 30, 90$.

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